If now we make use of the identity

$$(a'a''b')(a''a'''b''') = (a'a''a''')(a''b'b''') + (a'a''b''')(b'a''a'''),$$

and of the analogous ones

$$\begin{aligned}
(a'a''b'')(b'''a'a''') &= (a'a''a''')(a'b''b''') - (a'a''b''')(a'b''a'''),\\
(a'a'''b')(b''a''a''') &= -(a'a''a''')(b'b''a''') - (a'b''a''')(b'a''a'''),\\
(b'b''a')(a'''b''b''') &= -(b'b''b''')(a'b''a''') + (b'b''a''')(a'b''b'''),\\
(b'b''a'')(a'''b'b''') &= (b'b''b''')(b'a''a''') - (b'b''a''')(b'a''b'''),\\
(b'b'''a')(a''b''b''') &= (b'b''b''')(a'a''b''') - (b'a''b''')(a'b''b'''),
\end{aligned}$$

the above expansion reduces to the sum of squares

$$[(a'a''a''') - (a'b''b''') + (b'a''b''') - (b'b''a''')]^2$$
$$+ [(b'b''b''') - (b'a''a''') + (a'b''a'') - (a'a''b''')]^2.$$

If we represent by (A) the matrix of the a's, by $(-A)$ that of the $-a$'s, and similarly by (B) and $(-B)$ the matrices of the b's and $-b$'s respectively, the matrix of the determinant D may be represented symbolically as follows:

$$\begin{Vmatrix} (A) & (B) \\ (-B) & (A) \end{Vmatrix}.$$

It still remains to study determinants whose matrices are of the type

$$\begin{Vmatrix} (A) & (B) & (C)\cdots \\ (-B) & (A) & (D)\cdots \\ (-C) & (-D) & (A)\cdots \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \end{Vmatrix}.$$

August, 1914.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY.

Analytic Geometry of Space. By VIRGIL SNYDER and C. H. SISAM. Henry Holt and Co., New York, 1914. xi+289 pages.

It is probable that in no branch of elementary mathematics has there been such need of a good, teachable book as in the analytic geometry of space. Books on this subject, designed for two half-year courses, are strangely lacking. Charles Smith's book on solid geometry gives fine results when used with a small class of picked students because, like so many other text-books from England, it forces the student who would get anything from it to think, to remember and to coördinate many branches of elementary mathematics. In other words, the book is

not made up of a series of mechanical rules and illustrative examples, than which there is probably no influence more stultifying to a student's originality and power to attack a problem intelligently. Moreover, in such a book as Smith's, a student's mental capacity is treated with respect; as his mind becomes more virile, it is not nauseated by continual regurgitation of things once digested.

Smith's book, however, is lacking in many of the qualities and quantities which make the book under review a notable contribution to class-room literature.

Anyone sufficiently interested in analytic geometry to read this review can and probably will find a table of contents in the book itself. Therefore the reviewer will content himself with the privilege of calling attention to some of the exceptional features of the book.

Taken as a whole the book is noteworthy for its literary style. It is a delight to read it. Simple, forceful language is employed throughout, the theorems are models of clear expression and, when a paragraph is completed, its connection with the rest of the subject is apparent.

We are glad to note that, when the proof of some theorem has been standardized on account of its elegance, the authors have not felt obliged to bring forth an outrageous, new demonstration. The reviewer will never agree that desire for change is sufficient excuse to mangle a natural, tried and beautiful development.

The book is arranged so that its contents fall very naturally into materials for two courses, the first eight chapters constituting a course of some thirty-five lessons and the last six one of about fifty lessons.

The outstanding feature of the first eight chapters, and to the reviewer the finest thing in the book, is the natural and immediate introduction of certain geometric concepts which most authors seem to feel are better left out of a first course, such concepts as plane coördinates, homogeneous coördinates of points and planes, elements at infinity, imaginary elements, the absolute, circular points and isotropic planes.

The theorem that the locus of $Ax + By + Cz + D = 0$ (A, B, C, D real) is a plane is proved in a way which is very satisfying and, though a bit tedious, is very natural. The authors show that the locus satisfies three conditions that characterize a plane: (1) It contains three points not on a line; in fact, if $C \neq 0$, the three points $[0, 1, (-B - D)/C]$, $[1, 0, (-A - D)/C]$ and $(0, 0, -D/C)$. (2) It contains every point on any line joining two points on it. (3) It does not contain all points of space; in fact, it does not contain the point $[0, 0, (-D - C/C]$.

In dealing with the normal form of the equation of a plane and the perpendicular distance from a plane to a point, the authors use only the positive $\sqrt{A^2 + B^2 + C^2}$; they introduce the idea of a positive and negative side of the plane by the definition that (x_1, y_1, z_1) is on the positive side of the plane if $Ax_1 + By_1 + Cz_1 + D > 0$.

The point at infinity on a given line is introduced by saying that for every value of k, including $-a_4/b_4$, there is to correspond a point on the line $x_i = (a_i + b_ik)/(a_4 + b_4k)$ $(i = 1, 2, 3)$. Similarly, having introduced homogeneous coördinates as four numbers x', y', z', t' such that $x = x'/t'$, etc., a point is said to be defined even when $t' = 0$, and hence is a point at infinity.

The reviewer wishes especially to commend the early introduction of the so-called imaginary elements. He has always felt that the imagination and hence appreciation of a beginner in any sort of geometry is very greatly stimulated by an early consideration of these elements. It is well that he should be disillusioned at the start of the idea that the word *real* in geometry has any connection with the everyday word *reality*. The authors enforce this concept of imaginary elements by noting that the line determined by two conjugate points or planes is real, and similar theorems. They also furnish a fair list of examples.

The easy swing of the text is illustrated by the paragraph introducing the projecting cylinders of a curve: "The equation of the projecting cylinder of the curve of intersection of two surfaces $F(x, y, z) = 0$, $f(x, y, z) = 0$ on the plane $z = 0$ is independent of z. The equation of this cylinder may be obtained by eliminating z between the equations of the curve."

The absolute is introduced by proving that all spheres (it should be all non-composite spheres, to use the authors' terminology) intersect the plane at infinity in the same curve. The circular points and theorems connected with them, of course, enter very simply at this point. The authors err, I believe, in not making it sufficiently emphatic that all co-planar circles pass through the same circular points, after having defined the circular points as the two points in which any proper circle intersects the absolute.

The radical plane of two spheres is introduced as the ordinary plane of the composite sphere in $\lambda_1 S + \lambda_2 S' = 0$, where $S = 0$ and $S' = 0$ are two spheres, and it is then shown that this plane is the locus of the centers of spheres intersecting the two given spheres orthogonally. If we omit the consideration that this method of approach is general and illustrates the radical plane's relation to a system of spheres, we believe that several other ways of introducing the radical plane are more desirable, because less cumbersome and more directly interesting. The examples under this section are extremely well chosen.

Three chapters are devoted to an elementary discussion of quadrics. These surfaces are introduced by means of their simplest equations and are discussed with reference to their shape. I have yet to find an author who points out in his text how easy it is, by noting the exceptional sign and variable in these equations, to designate the type and position of the quadric. This ability to sketch a surface rapidly is a desideratum in many applications of analytic geometry.

In the treatment of the general Cartesian equation of quadrics, much stress is laid on the distinction between centers and vertices. If $F(x, y, z) = 0$ represents the given quadric, the lines and planes of centers and vertices are introduced by means of the rank of the system of planes $\partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$. I might say in passing that this chapter has nearly as many typographical errors as the preceding one has elegant figures and photographs.

The work on the discriminating cubic—one of the brightest spots in the analytic geometry of space—is particularly good, and the discussion of several of the special cases of quadrics is made very properly by means of illustrative examples. Little is gained in such cases by an involved, *general* discussion where difficulties

that do not really exist are apparently brought to light. In the examples connected with this chapter the authors have apparently gone to considerable trouble to get cases whose solutions involve complicated numbers and radicals.

The chapter on properties of quadrics is very concise. The treatment of circular sections deserves mention. It begins with the proof (given by two methods involving different principles) of the theorem: "Through each real, finite point of space pass six planes which intersect a given non-composite, non-spherical quadric in circles. If this quadric is not a surface of revolution nor a parabolic cylinder, these six planes are distinct; two are real and four imaginary. If the quadric is a surface of revolution or a parabolic cylinder, four of the planes are real and coincident and two imaginary." This theorem is then applied to standard forms of the quadric.

The point of view adopted in the ninth chapter and continued throughout the remainder of the book is quite different from that of the first part. Tetrahedral coördinates are here introduced by means of linear transformations; the principle of duality is used and more maturity in the student is demanded. As the authors state in the preface, the selection of subject matter in these last chapters is such as to be of greatest service for further study of algebraic geometry.

The road to an intelligent discussion of this subject matter is paved by two splendidly concise and clear chapters concerning such ideas as tetrahedral coördinates, duality, transformations of coördinates, cross ratio, invariance of the discriminant, lines on a quadric and polar theory. We recommend to anyone who has usually found the introduction of these concepts long and involved a careful perusal of chapters nine and ten of the book under review.

Chapter eleven, on linear systems of quadrics, is divided into four clearly differentiated parts, consisting respectively of pencils, bundles and webs of quadrics, and linear systems of rank r. The discussion of pencils is devoted to their classification into fifteen forms, arranged according to their *characteristic* (the symbol indicating the arrangement of roots in a given λ-discriminant). These forms are eventually tabulated, with a column for the curve of intersection of the two basic quadrics of the system. The culminating theorem of the section on webs of quadrics is that the points of the Kummer and Weddle surfaces are in one to one correspondence.

The reader will find the chapter on transformations one of the most satisfying of the book. It begins with two very illuminating and stimulating pages on the necessary and sufficient conditions that different elements are perpendicular, i. e., on poles and polars as to the absolute. Projective transformations are classified and tabulated in standard forms, fourteen in number, the determination of their characteristics and the loci of their invariant points being left as an exercise to the student. Birational transformations and the geometric constructions for some of them are mentioned very briefly, but in such an attractive manner as to give any student a taste for more.

The discussion of the algebraic surfaces

$$\Sigma \frac{n!}{\alpha!\beta!\gamma!\delta!} a_{\alpha\beta\gamma\delta} x_1{}^{\alpha} x_2{}^{\beta} x_3{}^{\gamma} x_4{}^{\delta} = 0$$

(α, β, γ, δ integers positive or zero and $\alpha + \beta + \gamma + \delta = n$) and algebraic space curves requires a distinctly more advanced knowledge of plane analytic geometry. When mastered the student will understand what things in general concern an investigator in algebraic curve theory and will have a good working knowledge of the main properties of space cubic and quartic curves.

After such a surfeit of praise as the above contains, the reviewer felt it his duty to look very carefully for something serious to criticize; but what he found seems very trivial. Of course the ubiquitous typographical errors are there and the figure used in finding the distance between two non-intersecting lines would seem to have been drawn by one who did not consult the text of that particular paragraph.

Polar, spherical, and cylindrical coördinates are very nicely introduced at the beginning, but are not mentioned again. If they have a place in the analytic geometry of space, that place should be at least visible.

Practically all the problems have the answers given and there are not enough of them incorrect to be of any pedagogical value. The reviewer would like to see the pages of this MONTHLY opened to a discussion of the relative pedagogical values of complete, partial and no lists of answers, and of answers all correct or partially incorrect.

The definition of analytic curves in the last short chapter on differential geometry is obviously made so as not to alarm the reader and is not precise. Incidentally it is made without any explanation of what an analytic function is.

For American students, at least, I feel very strongly that such a book would gain much in effectiveness if the first chapters were devoted to a concise discussion of those theorems on determinants and matrices that will be of service in the succeeding chapters. A teacher, having in mind what is to follow, has a great opportunity to make his students appreciate the great power and elegance of determinants.

Let me say in conclusion that, with its splendid style, its fine choice and arrangement of material and its pedagogical excellences, I believe this book one of the best contributions to American text-books made in recent years.

E. GORDON BILL.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

Note.—Of all the problems proposed between January 1, 1913, and January 1, 1915, satisfactory solutions for the following have not been received:

In Algebra, 406.
In Geometry, 427, 442, 446, and 454.
In Calculus, 339, 340, 342, 348, 353, 360, 363, and 368 to 375.
In Mechanics, 277–8–9, 287, and 291 to 301.
In Number Theory, 191–2, 196, 198, 205, 208–9–10–11, 214, 217, and 219 to 225.
Please give attention to these as well as to those proposed since January 1, 1915.